# The Post-Minkowskian Limit of $f(R)$-gravity

**Salvatore Capozziello · Arturo Stabile · Antonio Troisi**

**Abstract** We formally discuss the post-Minkowskian limit of $f(R)$-gravity without adopting conformal transformations but developing all the calculations in the original Jordan frame. It is shown that such an approach gives rise, in general, together with the standard massless graviton, to massive scalar modes whose masses are directly related to the analytic parameters of the theory. In this sense, the presence of massless gravitons only is a peculiar feature of General Relativity. This fact is never stressed enough and could have dramatic consequences in detection of gravitational waves. Finally the role of curvature stress-energy tensor of $f(R)$-gravity is discussed showing that it generalizes the so called Landau-Lifshitz tensor of General Relativity. The further degrees of freedom, giving rise to the massive modes, are directly related to the structure of such a tensor.

**Keywords** Alternative theories of gravity · Weak field limit · Gravitational waves

## 1 Introduction

Astrophysical observations of the last decade suggests the introduction of new ingredients in order to achieve a self-consistent picture of cosmos. In particular, the observation that Hubble flow is currently experiencing a speeding up has completely changed the approach to standard cosmology inducing to take into account theoretical approaches more general than the standard lore of General Relativity (GR).

S. Capozziello (✉)
Dipartimento di Scienze Fisiche, Università di Napoli "Federico II" and INFN sez. di Napoli Compl.
Univ. di Monte S. Angelo, Edificio G, Via Cinthia, 80126 Napoli, Italy
e-mail: capozzie@na.infn.it

A. Stabile
Dipartimento di Ingegneria, Universita' del Sannio and INFN sez. di Napoli, gruppo collegato
di Salerno, C-so Garibaldi, 80125 Benevento, Italy

A. Troisi
Dipartimento di Ingegneria Meccanica, Universita' di Salerno, Via Ponte Don Melillo,
84084 Fisciano (SA), Italy

The simplest explanation of such a cosmic acceleration requires to include the cosmological constant in the Friedmann-Robertson-Walker cosmology (Concordance Model). This ingredient gives rise to a negative pressure contribution needed to balance the standard matter attractive interaction. Although the Concordance Model represents the best fit model with respect to all samples of data coming from supernovae, large-scale structure and cosmic microwave radiation [1–3], several conceptual problems come out to theoretically define and to give an explanation to the observed value of the cosmological constant. Furthermore, assuming the existence of both dark energy and dark matter, we should find out new fundamental ingredients capable of giving account to almost 95% of the total amount of cosmic matter-energy.

Due to these difficulties, people have considered alternative approaches to GR that could be able to frame the observed late time acceleration and missing matter without introducing new ingredients. In this sense, higher order gravity [4–6] and, in particular, fourth order gravity represent an interesting scheme which could, potentially, address the problems.

Up to now, these theories have been investigated both at cosmological scale and in the weak field limit with significant results [7–21]. It has been shown that an accelerating late time behaviour can be easily recovered [22] and, in addition, it can be coherently related to an early time inflationary expansion [23, 24]. Furthermore, such an approach seems to deserve attention even at smaller scales. In fact, modifying the gravity action in favor of a non-linear Lagrangian in the Ricci scalar implies, in the Newtonian limit, corrections to the gravitational potential which can induce an astrophysical phenomenology interesting at galactic scales. In particular one can fit the rotation curves of spiral galaxies and the haloes of galactic clusters without the introduction of dark matter [25–30]. Besides, several of these extended models evade Solar System tests so they are not in conflict with positive experimental results of GR [31–33].

A relevant aspect of higher order gravity theories is that, in the post-Minkowskian limit (i.e. small fields and no prescriptions on the propagation velocity), the propagation of the gravitational fields turns out to be characterized by waves with both tensorial and scalar modes [34–36]. This issue represents a striking difference between GR and extended gravity since, in the standard Einstein scheme, only tensorial degrees of freedom are allowed. As matter of facts, the gravitational waves can represent a fundamental tool to discriminate between GR and alternative gravities [39, 40].

In this paper, we want to develop, formally, the post-Minkowskian limit of analytic $f(R)$-gravity models which, in our opinion, has never been pursued with accuracy stressing enough some peculiar points. As shown by the same authors for the Newtonian limit, we will show that it is different from the same limit of GR since massive modes naturally come out in the gravitational radiation. This occurrence has a deep meaning since points out that the presence of massless modes only is nothing else but the particular case of GR while massive and ghost modes are present in general [35, 36].

The layout of the paper is the following. In Sect. 2, we discuss the post-Minkowskian limit of $f(R)$-gravity. Considerations on gravitational wave massive modes are developed in Sect. 3. Section 4 is devoted to the discussion of the role of the stress-energy tensor in $f(R)$-gravity. Concluding remarks are drawn in Sect. 5.

## 2 The Post-Minkowskian Limit of $f(R)$-gravity

Any theory of gravity has to be discussed in the weak field limit approximation. This "prescription" is needed to test if the given theory is consistent with the well-established Newtonian Theory and with the Special Relativity as soon as the gravitational field is weak or is

almost null. Both requirements are fulfilled by GR and then they can be considered two possible paradigms to confront a given theory, at least in the weak field limit, with GR itself. In [41, 42], the Newtonian limit of $f(R)$-gravity is investigated always remaining in the Jordan frame [37, 38]. From our point of view, this is important since, by perturbatively approximating a field, some conformal features could be lost. Here we want to derive, formally, the post-Minkowskian limit of $f(R)$-gravity.

The post-Minkowskian limit of any theory of gravity arises when the regime of small field is considered without any prescription on the propagation of the field. This case has to be clearly distinguished with respect to the Newtonian limit which, differently, requires both the small velocity and the weak field approximations. Often, in literature, such a distinction is not clearly remarked and several cases of pathological analysis can be accounted. The post-Minkowskian limit of GR gives rise to massless gravitational waves. An analogous study can be pursued considering, instead of the Hilbert-Einstein Lagrangian linear in the Ricci scalar $R$, a general function $f(R)$. The only assumption that we are going to do is that $f(R)$ is an analytic function. The gravitational action is then

$$\mathcal{A} = \int d^4x \sqrt{-g}\big[f(R) + \mathcal{X}\mathcal{L}_m\big], \tag{1}$$

where $\mathcal{X} = \frac{16\pi G}{c^4}$ is the coupling, $\mathcal{L}_m$ is the standard matter Lagrangian and $g$ is the determinant of the metric. The field equations, in metric formalism, read[1]

$$f'R_{\mu\nu} - \frac{1}{2}fg_{\mu\nu} - f'_{;\mu\nu} + g_{\mu\nu}\Box_g f' = \frac{\mathcal{X}}{2}T_{\mu\nu}, \tag{2}$$

$$3\Box f' + f'R - 2f = \frac{\mathcal{X}}{2}T, \tag{3}$$

with $T_{\mu\nu} = \frac{-2}{\sqrt{-g}}\frac{\delta(\sqrt{-g}\mathcal{L}_m)}{\delta g^{\mu\nu}}$ the energy momentum tensor of matter ($T$ is the trace), $f' = \frac{df(R)}{dR}$ and $\Box_g = {}_{;\sigma}{}^{;\sigma}$. We adopt a $(+, -, -, -)$ signature, while the conventions for Ricci's tensor is $R_{\mu\nu} = R^\sigma{}_{\mu\sigma\nu}$ and $R^\alpha{}_{\beta\mu\nu} = \Gamma^\alpha_{\beta\nu,\mu} + \cdots$ for the Riemann tensor, where

$$\Gamma^\mu_{\alpha\beta} = \frac{1}{2}g^{\mu\sigma}(g_{\alpha\sigma,\beta} + g_{\beta\sigma,\alpha} - g_{\alpha\beta,\sigma}) \tag{4}$$

are the Christoffel symbols of the $g_{\mu\nu}$ metric. Actually, in order to perform a post-Minkowskian limit of field equations, one has to perturb (2) on the Minkowski background $\eta_{\mu\nu}$. In such a case the invariant metric element becomes

$$ds^2 = g_{\sigma\tau}dx^\sigma dx^\tau = (\eta_{\sigma\tau} + h_{\sigma\tau})dx^\sigma dx^\tau \tag{5}$$

with $h_{\mu\nu}$ small ($O(h)^2 \ll 1$). We assume that the $f(R)$-Lagrangian is analytic (i.e. Taylor expandable) in term of the Ricci scalar, which means that

$$f(R) = \sum_n \frac{f^n(R_0)}{n!}(R - R_0)^n \simeq f_0 + f'_0 R + \frac{1}{2}f''_0 R^2 + \cdots. \tag{6}$$

The flat-Minkowski background is recovered for $R = R_0 \simeq 0$.

---

[1]All considerations are developed here in metric formalism.

Field equations (2), at the first order of approximation in term of the perturbation [43], become:

$$f_0'\left[R_{\mu\nu}^{(1)} - \frac{R^{(1)}}{2}\eta_{\mu\nu}\right] - f_0''\left[R_{,\mu\nu}^{(1)} - \eta_{\mu\nu}\Box R^{(1)}\right] = \frac{\mathcal{X}}{2}T_{\mu\nu}^{(0)} \tag{7}$$

where $f_0' = \frac{df}{dR}|_{R=0}$, $f_0'' = \frac{d^2f}{dR^2}|_{R=0}$ and $\Box = ,_\sigma{}^{,\sigma}$ that is now the standard d'Alembert operator of flat space-time. From the zero-order of (2), one gets $f(0) = 0$, while $T_{\mu\nu}$ is fixed at zero-order in (7) since, in this perturbation scheme, the first order on Minkowski space has to be connected with the zero order of the standard matter energy momentum tensor.[2] The explicit expressions of the Ricci tensor and scalar, at the first order in the metric perturbation, read

$$\begin{cases} R_{\mu\nu}^{(1)} = h_{(\mu,\nu)\sigma}^\sigma - \frac{1}{2}\Box h_{\mu\nu} - \frac{1}{2}h_{,\mu\nu}, \\ R^{(1)} = h_{\sigma\tau}{}^{,\sigma\tau} - \Box h \end{cases} \tag{8}$$

with $h = h^\sigma{}_\sigma$. Equation (7) can be written in a more suitable form by introducing the constant $\xi = -\frac{f_0''}{f_0'}$, that is

$$h_{(\mu,\nu)\sigma}^\sigma - \frac{1}{2}\Box h_{\mu\nu} - \frac{1}{2}h_{,\mu\nu} - \frac{1}{2}(h_{\sigma\tau}{}^{,\sigma\tau} - \Box h)\eta_{\mu\nu} + \xi(\partial_{\mu\nu}^2 - \eta_{\mu\nu}\Box)(h_{\sigma\tau}{}^{,\sigma\tau} - \Box h) = \frac{\mathcal{X}}{2f_0'}T_{\mu\nu}^{(0)}. \tag{9}$$

By choosing the transformation $\tilde{h}_{\mu\nu} = h_{\mu\nu} - \frac{h}{2}\eta_{\mu\nu}$ and the gauge condition $\tilde{h}^{\mu\nu}{}_{,\mu} = 0$, one obtains that field equations and the trace equation, respectively, read[3]

$$\begin{cases} \Box\tilde{h}_{\mu\nu} + \xi(\eta_{\mu\nu}\Box - \partial_{\mu\nu}^2)\Box\tilde{h} = -\frac{\mathcal{X}}{f_0'}T_{\mu\nu}^{(0)}, \\ \Box\tilde{h} + 3\xi\Box^2\tilde{h} = -\frac{\mathcal{X}}{f_0'}T^{(0)}. \end{cases} \tag{10}$$

In order to derive the analytic solutions of (10), we can adopt a momentum- description. This approach simplifies the equations and allows to fix the physical properties of the problem. In such a scheme, we have:

$$\begin{cases} k^2\tilde{h}_{\mu\nu}(k) + \xi(k_\mu k_\nu - k^2\eta_{\mu\nu})k^2\tilde{h}(k) = \frac{\mathcal{X}}{f_0'}T_{\mu\nu}^{(0)}(k), \\ k^2\tilde{h}(k)(1 - 3\xi k^2) = \frac{\mathcal{X}}{f_0'}T^{(0)}(k), \end{cases} \tag{11}$$

where

$$\begin{cases} \tilde{h}_{\mu\nu}(k) = \int \frac{d^4x}{(2\pi)^2}\tilde{h}_{\mu\nu}(x)e^{-ikx}, \\ T_{\mu\nu}^{(0)}(k) = \int \frac{d^4x}{(2\pi)^2}T_{\mu\nu}^{(0)}(x)e^{-ikx} \end{cases} \tag{12}$$

---

[2]This formalism descends from the theoretical setting of Newtonian mechanics which requires the appropriate scheme of approximation when obtained from a more general relativistic theory. This scheme coincides with a gravity theory analyzed at the first order of perturbation in the curved spacetime metric.

[3]The gauge transformation is $h'_{\mu\nu} = h_{\mu\nu} - \zeta_{\mu,\nu} - \zeta_{\nu,\mu}$ when we perform a coordinate transformation as $x'^\mu = x^\mu + \zeta^\mu$ with $O(\zeta^2) \ll 1$. To obtain the gauge and the validity of the field equations for both perturbation $h_{\mu\nu}$ and $\tilde{h}_{\mu\nu}$, the $\zeta_\mu$ have to satisfy the harmonic condition $\Box\zeta^\mu = 0$.

are the Fourier transforms of the perturbation $\tilde{h}_{\mu\nu}(x)$ and of the matter tensor $T_{\mu\nu}^{(0)}$. We have defined, as usual, $kx = \omega t - \mathbf{k} \cdot \mathbf{x}$ and $k^2 = \omega^2 - \mathbf{k}^2$; $\tilde{h}(k)$ and $T^{(0)}(k)$ are the traces of $\tilde{h}_{\mu\nu}(k)$ and $T_{\mu\nu}^{(0)}(k)$. In the momentum space, one can easily recognize the solutions of (11); $\tilde{h}_{\mu\nu}(k)$ turns out to be

$$\tilde{h}_{\mu\nu}(k) = \frac{\mathcal{X}}{f_0'} \frac{T_{\mu\nu}^{(0)}(k)}{k^2} + \frac{\xi \mathcal{X}}{f_0'} \frac{k^2 \eta_{\mu\nu} - k_\mu k_\nu}{k^2 (1 - 3\xi k^2)} T^{(0)}(k), \tag{13}$$

which fulfills the condition $\tilde{h}^{\mu\nu}_{,\mu} = 0$ (that is $\tilde{h}^{\mu\nu}(k) k_\mu = 0$). The perturbation variable $h_{\mu\nu}(k)$ can be obtained by inverting the relation with the tilded variables. In particular, by inserting the new stress-energy tensor $S_{\mu\nu}^{(0)}(k) = T_{\mu\nu}^{(0)}(k) - \frac{1}{2}\eta_{\mu\nu} T^{(0)}(k)$ with the trace $S^{(0)}(k) = \eta^{\mu\nu} S_{\mu\nu}^{(0)}(k)$, one obtains:

$$h_{\mu\nu}(k) = \frac{\mathcal{X}}{f_0'} \frac{S_{\mu\nu}^{(0)}(k)}{k^2} + \frac{\xi \mathcal{X}}{2 f_0'} \frac{k^2 \eta_{\mu\nu} + 2 k_\mu k_\nu}{k^2 (1 - 3\xi k^2)} S^{(0)}(k), \tag{14}$$

which represents a wave-like solution, in the momentum space, with a massless and a massive contributions. The massive term is due to the pole in the denominator of the second term: the *mass* is directly related with the physical properties of the pole itself and, thanks to the parameter $\xi$, depends on the analytic form of the model (i.e. $f_0'$ and $f_0''$). The wavelike solution in the configuration space is obtained by the inverse Fourier transform of $h_{\mu\nu}(k)$.

## 3 Massive Modes in Gravitational Waves

The presence of the massive term is a feature emerging from the intrinsic non-linearity of $f(R)$-gravity. Specifically, it is related to the fact that $f_0'' \neq 0$, which is zero in GR where $f(R) = R$. This means that massless states are nothing else but a particular case among the gravitational theories. A similar situation emerges also in the Newtonian limit: the Newton potential is recovered only as the weak field limit of GR. In general, Yukawa-like corrections, and then characteristic interaction lengths, are present [41]

Some considerations are in order at this point. It is worth noticing that field equations (2) can be written putting in evidence the Einstein tensor in the l.h.s. [7–9]. In such a case, higher than second order differential contributions can be considered as a sources in the r.h.s. as well as the energy-momentum tensor of standard matter:

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = T_{\mu\nu}^{(curv)} + T_{\mu\nu}^{(m)}, \tag{15}$$

where

$$\begin{cases} T_{\mu\nu}^{(m)} = \frac{T_{\mu\nu}}{f'(R)}, \\ T_{\mu\nu}^{(curv)} = \frac{1}{2} g_{\mu\nu} \frac{f(R) - f'(R) R}{f'(R)} + \frac{f'(R)_{;\mu\nu} - g_{\mu\nu} \Box_g f'(R)}{f'(R)}. \end{cases} \tag{16}$$

Actually, if we consider the perturbed metric (5) and develop the Einstein tensor up to the first order in perturbations, we have

$$G_{\mu\nu} \sim G_{\mu\nu}^{(1)} = h^\sigma_{(\mu,\nu)\sigma} - \frac{1}{2} \Box h_{\mu\nu} - \frac{1}{2} h_{,\mu\nu} - \frac{1}{2} (h_{\sigma\tau}^{,\sigma\tau} - \Box h) \eta_{\mu\nu} \tag{17}$$

while the curvature stress-energy tensor gives the contributions

$$T_{\mu\nu}^{(curv)} \sim \xi(\partial_{\mu\nu}^2 - \eta_{\mu\nu}\Box)(h_{\sigma\tau}{}^{,\sigma\tau} - \Box h). \tag{18}$$

This expression allows to recognize that, in the space of momenta, such a quantity will be responsible of the pole-like term which implies the introduction of a massive degree of freedom into the particle spectrum of gravity. In fact, inserting these two expressions into the field equations (15) and considering (8), we obtain the solution:

$$\Box h_{\mu\nu}(x) = -\frac{\mathcal{X}}{f_0'}\big[S_{\mu\nu}^{(0)}(x) + \Sigma_{\mu\nu}^{\xi}(x)\big], \tag{19}$$

where $\Sigma_{\mu\nu}^{\xi}(x)$ is related to the curvature stress-energy tensor and is defined as

$$\Sigma_{\mu\nu}^{\xi}(x) = \frac{\xi}{2} \int \frac{d^4k}{(2\pi)^2} \frac{k^2\eta_{\mu\nu} + 2k_\mu k_\nu}{1 - 3\xi k^2} S^{(0)}(k)e^{ikx}. \tag{20}$$

The general solution for the metric perturbation $h_{\mu\nu}(x)$, when the field equations are (15), can be rewritten as

$$h_{\mu\nu}(x) = \frac{\mathcal{X}}{f_0'} \int \frac{d^4k}{(2\pi)^2} \frac{S_{\mu\nu}^{(0)}(k)}{k^2} e^{ikx} + \frac{\xi\mathcal{X}}{2f_0'} \int \frac{d^4k}{(2\pi)^2} \frac{k^2\eta_{\mu\nu} + 2k_\mu k_\nu}{k^2(1 - 3\xi k^2)} S^{(0)}(k)e^{ikx}, \tag{21}$$

where the second pole-like term is present. In vacuum (i.e. $T_{\mu\nu}^{(m)} = 0$), (10) become

$$\begin{cases} k^2[\tilde{h}_{\mu\nu}(k) + \xi(k_\mu k_\nu - k^2\eta_{\mu\nu})\tilde{h}(k)] = 0, \\ k^2\tilde{h}(k)(1 - 3\xi k^2) = 0 \end{cases} \tag{22}$$

showing that allowed solutions are of two types, i.e.:

$$\begin{cases} \omega = \pm|\mathbf{k}|, \\ h_{\mu\nu}(x) = \int \frac{d^4k}{(2\pi)^2} h_{\mu\nu}(k)e^{ikx} \quad \text{with } h(k) = 0, \end{cases} \tag{23}$$

and

$$\begin{cases} \omega = \pm\sqrt{\mathbf{k}^2 + \frac{1}{3\xi}}, \\ h_{\mu\nu}(x) = -\int \frac{d^4k}{(2\pi)^2} [\frac{\eta_{\mu\nu} + 6\xi k_\mu k_\nu}{6}] h(k)e^{ikx} \quad \text{with } h(k) \neq 0. \end{cases} \tag{24}$$

It is evident, that the first solution represents a massless graviton according to the standard prescriptions of GR while the second one gives a massive degree of freedom with $m^2 = (3\xi)^{-1} = -\frac{f_0'}{3f_0''}$. Thanks to this property, we can rewrite (10) introducing a scalar field $\phi = \Box\tilde{h}$ so that the general system can be rearranged in the following way

$$\begin{cases} \Box\tilde{h}_{\mu\nu} = -\frac{\mathcal{X}}{f_0'} T_{\mu\nu}^{(0)} + [\frac{\partial_{\mu\nu}^2 - \eta_{\mu\nu}\Box}{3m^2}]\phi, \\ (\Box + m^2)\phi = -\frac{\mathcal{X}}{f_0'} m^2 T^{(0)}, \end{cases} \tag{25}$$

which suggests that the higher order contributions act, in the post-Minkowskian limit, as a massive scalar field whose mass depends on the derivatives $f'(R)$ and $f''(R)$, calculated on the unperturbed background metric.

The massive mode is directly related to the coefficients of the Taylor expansion and it is interesting to note that they dermine also the value of the Yukawa correction in the Newtonian approximation [41, 42]. On the other hand, it is straightforward to see that massive modes are directly related to the non-trivial structure of the trace equation as it is easy to see from (3). In GR, the Ricci scalar is univocally fixed being $R = 0$ in vacuum and $R \propto \rho$ in presence of matter, where $\rho$ is the matter-energy density.

## 4 The Stress-energy Tensor in $f(R)$-gravity and the Gravitational Radiation

As we have seen, higher order theories of gravity introduce further degrees of freedom which can be taken into account by defining an additional "curvature source term" in the r.h.s. of field equations. This quantity behaves as an effective stress-energy tensor that can characterize the energy loss due to the gravitational radiation. Although the procedure to calculate the stress-energy tensor of the gravitational field in GR is often debated, one can extend the formalism to more general theories and obtain this quantity by varying the gravitational Lagrangian. In GR, this quantity is a pseudo-tensor and is typically referred to as the Landau-Lifshitz energy-momentum tensor [44].

The calculations of GR need to be extended when dealing with higher order gravity. In the case of $f(R)$-gravity, we have

$$
\begin{aligned}
\delta \int d^4x \sqrt{-g}\, f(R) &= \delta \int d^4x \mathcal{L}(g_{\mu\nu}, g_{\mu\nu,\rho}, g_{\mu\nu,\rho\sigma}) \\
&\approx \int d^4x \left( \frac{\partial \mathcal{L}}{\partial g_{\rho\sigma}} - \partial_\lambda \frac{\partial \mathcal{L}}{\partial g_{\rho\sigma,\lambda}} + \partial^2_{\lambda\xi} \frac{\partial \mathcal{L}}{\partial g_{\rho\sigma,\lambda\xi}} \right) \delta g_{\rho\sigma} \\
&\doteq \int d^4x \sqrt{-g}\, H^{\rho\sigma} \delta g_{\rho\sigma} = 0.
\end{aligned}
\tag{26}
$$

The Euler-Lagrange equations are then

$$
\frac{\partial \mathcal{L}}{\partial g_{\rho\sigma}} - \partial_\lambda \frac{\partial \mathcal{L}}{\partial g_{\rho\sigma,\lambda}} + \partial^2_{\lambda\xi} \frac{\partial \mathcal{L}}{\partial g_{\rho\sigma,\lambda\xi}} = 0,
\tag{27}
$$

which coincide with the field equations (2) in vacuum. Actually, even in the case of more general theories, it is possible to define an energy-momentum tensor that turns out to be defined as follows:

$$
t^\lambda_\alpha = \frac{1}{\sqrt{-g}} \left[ \left( \frac{\partial \mathcal{L}}{\partial g_{\rho\sigma,\lambda}} - \partial_\xi \frac{\partial \mathcal{L}}{\partial g_{\rho\sigma,\lambda\xi}} \right) g_{\rho\sigma,\alpha} + \frac{\partial \mathcal{L}}{\partial g_{\rho\sigma,\lambda\xi}} g_{\rho\sigma,\xi\alpha} - \delta^\lambda_\alpha \mathcal{L} \right].
\tag{28}
$$

This quantity, together with the energy-momentum tensor of matter $T_{\mu\nu}$, satisfies a conservation law as required by the Bianchi identities. In fact, in presence of matter, one has $H_{\mu\nu} = \frac{\chi}{2} T_{\mu\nu}$, and then

$$
(\sqrt{-g}\, t^\lambda_\alpha)_{,\lambda} = -\sqrt{-g}\, H^{\rho\sigma} g_{\rho\sigma,\alpha} = -\frac{\chi}{2} \sqrt{-g}\, T^{\rho\sigma} g_{\rho\sigma,\alpha} = -\chi (\sqrt{-g}\, T^\lambda_\alpha)_{,\lambda},
\tag{29}
$$

and, as a consequence,

$$[\sqrt{-g}(t_\alpha^\lambda + \mathcal{X}T_\alpha^\lambda)]_{,\lambda} = 0 \tag{30}$$

that is the conservation law given by the Bianchi identities. We can now write the expression of the energy-momentum tensor $t_\alpha^\lambda$ in term of the gravity action $f(R)$ and its derivatives:

$$t_\alpha^\lambda = f'\left\{\left[\frac{\partial R}{\partial g_{\rho\sigma,\lambda}} - \frac{1}{\sqrt{-g}}\partial_\xi\left(\sqrt{-g}\frac{\partial R}{\partial g_{\rho\sigma,\lambda\xi}}\right)\right]g_{\rho\sigma,\alpha} + \frac{\partial R}{\partial g_{\rho\sigma,\lambda\xi}}g_{\rho\sigma,\xi\alpha}\right\}$$
$$- f''R_{,\xi}\frac{\partial R}{\partial g_{\rho\sigma,\lambda\xi}}g_{\rho\sigma,\alpha} - \delta_\alpha^\lambda f. \tag{31}$$

It is worth noticing that $t_\alpha^\lambda$ is a non-covariant quantity in GR while its generalization, in fourth order gravity, turns out to satisfy the covariance prescription of standard tensors (see also [4–6]). On the other hand, such an expression reduces to the Landau-Lifshitz energy-momentum tensor of GR as soon as $f(R) = R$, that is

$$t_{\alpha\,|\text{GR}}^\lambda = \frac{1}{\sqrt{-g}}\left(\frac{\partial \mathcal{L}_{\text{GR}}}{\partial g_{\rho\sigma,\lambda}}g_{\rho\sigma,\alpha} - \delta_\alpha^\lambda\mathcal{L}_{\text{GR}}\right) \tag{32}$$

where the GR Lagrangian has been considered in its effective form, i.e. the symmetric part of the Ricci tensor, which effectively leads to the equations of motion, that is

$$\mathcal{L}_{\text{GR}} = \sqrt{-g}g^{\mu\nu}(\Gamma_{\mu\sigma}^\rho\Gamma_{\rho\nu}^\sigma - \Gamma_{\mu\nu}^\sigma\Gamma_{\sigma\rho}^\rho). \tag{33}$$

It is important to stress that the definition of the energy-momentum tensor in GR and in $f(R)$-gravity are different. This discrepancy is due to the presence, in the second case, of higher than second order differential terms that cannot be discarded by means of a boundary integration as it is done in GR. We have noticed that the effective Lagrangian of GR turns out to be the symmetric part of the Ricci scalar since the second order terms, present in the definition of $R$, can be removed by means of integration by parts.

   On the other hand, an analytic $f(R)$-Lagrangian can be recast, at linear order, as $f \sim f_0'R + \mathcal{F}(R)$, where the function $\mathcal{F}$ satisfies the condition: $\lim_{R\to0}\mathcal{F} \to R^2$. As a consequence, one can rewrite the explicit expression of $t_\alpha^\lambda$ as:

$$t_\alpha^\lambda = f_0't_{\alpha\,|\text{GR}}^\lambda + \mathcal{F}'\left\{\left[\frac{\partial R}{\partial g_{\rho\sigma,\lambda}} - \frac{1}{\sqrt{-g}}\partial_\xi\left(\sqrt{-g}\frac{\partial R}{\partial g_{\rho\sigma,\lambda\xi}}\right)\right]g_{\rho\sigma,\alpha} + \frac{\partial R}{\partial g_{\rho\sigma,\lambda\xi}}g_{\rho\sigma,\xi\alpha}\right\}$$
$$- \mathcal{F}''R_{,\xi}\frac{\partial R}{\partial g_{\rho\sigma,\lambda\xi}}g_{\rho\sigma,\alpha} - \delta_\alpha^\lambda\mathcal{F}. \tag{34}$$

The general expression of the Ricci scalar, obtained by splitting its linear ($R^*$) and quadratic ($\bar{R}$) parts once a perturbed metric (5) is considered, is

$$R = g^{\mu\nu}(\Gamma_{\mu\nu,\rho}^\rho - \Gamma_{\mu\rho,\nu}^\rho) + g^{\mu\nu}(\Gamma_{\sigma\rho}^\rho\Gamma_{\mu\nu}^\sigma - \Gamma_{\rho\mu}^\sigma\Gamma_{\nu\sigma}^\rho) = R^* + \bar{R} \tag{35}$$

(notice that $\mathcal{L}_{\text{GR}} = -\sqrt{-g}\bar{R}$). In the case of GR $t_{\alpha\,|\text{GR}}^\lambda$, the Landau-Lifshitz tensor presents a first non-vanishing term at order $h^2$. A similar result can be obtained in the case of $f(R)$-

gravity. In fact, taking into account (34), one obtains that, at the lower order, $t_\alpha^\lambda$ reads:

$$
t_\alpha^\lambda \sim t_{\alpha\,|h^2}^\lambda = f_0' t_{\alpha\,|\mathrm{GR}}^\lambda + f_0'' R^* \left[ \left( -\partial_\xi \frac{\partial R^*}{\partial g_{\rho\sigma,\lambda\xi}} \right) g_{\rho\sigma,\alpha} + \frac{\partial R^*}{\partial g_{\rho\sigma,\lambda\xi}} g_{\rho\sigma,\xi\alpha} \right]
$$

$$
- f_0'' R_{,\xi}^* \frac{\partial R^*}{\partial g_{\rho\sigma,\lambda\xi}} g_{\rho\sigma,\alpha} - \frac{1}{2} f_0'' \delta_\alpha^\lambda R^{*2}
$$

$$
= f_0' t_{\alpha\,|\mathrm{GR}}^\lambda + f_0'' \left[ R^* \left( \frac{\partial R^*}{\partial g_{\rho\sigma,\lambda\xi}} g_{\rho\sigma,\xi\alpha} - \frac{1}{2} R^* \delta_\alpha^\lambda \right) - \partial_\xi \left( R^* \frac{\partial R^*}{\partial g_{\rho\sigma,\lambda\xi}} \right) g_{\rho\sigma,\alpha} \right]. \quad (36)
$$

Considering the perturbed metric (5), we have $R^* \sim R^{(1)}$, where $R^{(1)}$ is defined as in (8). In terms of $h$ and $\eta$, we get

$$
\begin{cases}
\frac{\partial R^*}{\partial g_{\rho\sigma,\lambda\xi}} \sim \frac{\partial R^{(1)}}{\partial h_{\rho\sigma,\lambda\xi}} = \eta^{\rho\lambda}\eta^{\sigma\xi} - \eta^{\lambda\xi}\eta^{\rho\sigma}, \\[2mm]
\frac{\partial R^*}{\partial g_{\rho\sigma,\lambda\xi}} g_{\rho\sigma,\xi\alpha} \sim h^{\lambda\xi}{}_{,\xi\alpha} - h^{,\lambda}{}_\alpha.
\end{cases}
\quad (37)
$$

Clearly, the first significant term in (36) is of second order in the perturbation expansion. We can now write the expression of the energy-momentum tensor explicitly in term of the perturbation $h$; it is

$$
t_\alpha^\lambda \sim f_0' t_{\alpha\,|\mathrm{GR}}^\lambda + f_0'' \left\{ (h^{\rho\sigma}{}_{,\rho\sigma} - \Box h) \left[ h^{\lambda\xi}{}_{,\xi\alpha} - h^{,\lambda}{}_\alpha - \frac{1}{2}\delta_\alpha^\lambda (h^{\rho\sigma}{}_{,\rho\sigma} - \Box h) \right] \right.
$$

$$
\left. - h^{\rho\sigma}{}_{,\rho\sigma\xi} h^{\lambda\xi}{}_{,\alpha} + h^{\rho\sigma}{}_{,\rho\sigma}{}^\lambda h_{,\alpha} + h^{\lambda\xi}{}_{,\alpha}\Box h_{,\xi} - \Box h^{,\lambda} h_{,\alpha} \right\}. \quad (38)
$$

Considering the tilded perturbation metric $\tilde{h}_{\mu\nu}$, the more compact form

$$
t_{\alpha\,|f}^\lambda = \frac{1}{2}\left[ \frac{1}{2}\tilde{h}^{,\lambda}{}_\alpha \Box\tilde{h} - \frac{1}{2}\tilde{h}_{,\alpha}\Box\tilde{h}^{,\lambda} - \tilde{h}^\lambda{}_{\sigma,\alpha}\Box\tilde{h}^{,\sigma} - \frac{1}{4}(\Box\tilde{h})^2 \delta_\alpha^\lambda \right], \quad (39)
$$

is achieved. As matter of facts, the energy-momentum tensor of the gravitational field, which expresses the energy transport during the propagation, has a natural generalization in the case of $f(R)$-gravity. We have adopted here the Landau-Lifshitz definition but other approaches can be taken into account [45]. The general definition of $t_\alpha^\lambda$, obtained above, consists of a sum of a GR contribution plus a term coming from $f(R)$-gravity:

$$
t_\alpha^\lambda = f_0' t_{\alpha\,|\mathrm{GR}}^\lambda + f_0'' t_{\alpha\,|f}^\lambda. \quad (40)
$$

However, as soon as $f(R) = R$, we obtains $t_\alpha^\lambda = t_{\alpha\,|\mathrm{GR}}^\lambda$. As a final remark, it is worth noticing that massive modes of gravitational field come out from $t_{\alpha\,|f}^\lambda$ since $\Box\tilde{h}$ can be considered an effective scalar field moving in a potential: $t_\alpha^\lambda$, in this case, represents a transport tensor.

## 5 Concluding Remarks

In this paper, we have formally studied the post-Minkowskian limit of $f(R)$-gravity developing all the calculations in the Jordan frame. The main result is that, beside standard massless modes of GR, further massive modes emerge and they are directly determined by

the analytic parameters of $f(R)$-gravity, that is the coefficients $f_0'$ and $f_0''$ of the Taylor expansion. This fact is extremely relevant since it does not depend on the considered $f(R)$-model but it is a general feature that can be enunciated in the following way: *Massless gravitons are a peculiar characteristic of GR while extended or alternative theories have, in general, further massive or ghost states* [35, 36]. It is worth noticing that several indications in this sense are present in literature [34, 46, 47] but their relevance, from an experimental viewpoint, has never been stressed enough.

On the other hand, a similar result comes out also in the Newtonian limit of the same theories: Yukawa-like corrections to the gravitational potential emerge in general and they are absent only in the case of GR. It is interesting to note that also the characteristic lengths of such corrections are related to $f_0'$ and $f_0''$ as shown in [41]. Also in this case, the Newtonian potential, coming from the weak field limit of GR, is only a particular case.

These results pose interesting problems related to the validity of GR at all scales. It seems that it works very well at local scales (Solar System) where effects of further gravitational degrees of freedom cannot be detected. As soon as one is investigating larger scales, as those of galaxies, clusters of galaxies, etc., further corrections have to be introduced in order to explain both astrophysical large-scale dynamics [22, 25–30] and cosmic evolution [7–9, 31]. Alternatively, huge amounts of dark matter and dark energy have to be invoked to explain the phenomenology, but, up today there are no final answer for these new constituents at fundamental level. Furthermore, the fact that, up to now, only massless gravitational waves have been investigated could be a shortcoming preventing the possibility to find out other forms of gravitational radiation. Tests in this sense could come, for example, from the stochastic background of gravitational waves where massive modes could play a crucial role in the cosmic background spectrum [48, 49].

## References

1. Padmanabhan, T.: Phys. Rep. **380**, 235 (2003)
2. Peebles, P.J.E., Ratra, B.: Rev. Mod. Phys. **75**, 559 (2003)
3. Sahni, V.: Lect. Notes Phys. **653**, 141 (2004)
4. Nojiri, S., Odintsov, S.D.: Int. J. Geom. Methods Mod. Phys. **4**, 115 (2007)
5. Capozziello, S., Francaviglia, M.: Gen. Relativ. Gravit. **40**, 357 (2008)
6. Sotiriou, T.P., Faraoni, V.: arXiv:0805.1726 [gr-qc] (2008)
7. Capozziello, S.: Int. J. Mod. Phys. D **11**, 483 (2002)
8. Capozziello, S., Carloni, S., Troisi, A.: Rec. Res. Dev. Astron. Astrophys. **1**, 625 (2003). arXiv:astro-ph/0303041
9. Capozziello, S., Cardone, V.F., Carloni, S., Troisi, A.: Int. J. Mod. Phys. D **12**, 1969 (2003)
10. Carroll, S.M., Duvvuri, V., Trodden, M., Turner, M.S.: Phys. Rev. D **70**, 043528 (2004)
11. Nojiri, S., Odintsov, S.D.: Phys. Lett. B **576**, 5 (2003)
12. Nojiri, S., Odintsov, S.D.: Phys. Rev. D **68**, 12352 (2003)
13. Capozziello, S., Cardone, V.F., Troisi, A.: Phys. Rev. D **71**, 043503 (2005)
14. Carloni, S., Dunsby, P.K.S., Capozziello, S., Troisi, A.: Class. Quantum Gravity **22**, 4839 (2005)
15. Cognola, G., Elizalde, E., Nojiri, S., Odintsov, S.D., Sebastiani, L., Zerbini, S.: Phys. Rev. D **77**, 046009 (2008)
16. Cognola, G., Zerbini, S.: Int. J. Theor. Phys. **47**, 3186 (2008)
17. Olmo, G.J.: Phys. Rev. Lett. **95**, 261102 (2005)
18. Capozziello, S., Troisi, A.: Phys. Rev. D **72**, 044022 (2005)
19. Capozziello, S., Stabile, A., Troisi, A.: Mod. Phys. Lett. A **21**, 2291 (2006)
20. Sotiriou, T.P.: Gen. Relativ. Gravit. **38**, 1407 (2006)
21. Chiba, T.: Phys. Lett. B **575**, 1 (2005)
22. Saffari, R., Rahvar, S.: Phys. Rev. D **77**, 104028 (2008)
23. Nojiri, S., Odintsov, S.D.: Phys. Rev. D **77**, 026007 (2008)
24. Nojiri, S., Odintsov, S.D., Tretyakov, P.V.: Prog. Theor. Phys. Suppl. **172**, 81 (2008)

25. Capozziello, S., Cardone, V.F., Troisi, A.: JCAP **0608**, 001 (2006)
26. Capozziello, S., Cardone, V.F., Troisi, A.: Mon. Not. R. Astron. Soc. **375**, 1423 (2007)
27. Sobouti, Y.: Astron. Astrophys. **464**, 921 (2007)
28. Mendoza, S., Rosas-Guevara, Y.M.: Astron. Astrophys. **472**, 367 (2007)
29. Boehmer, Ch.G., Harko, T., Lobo, F.S.N.: Astropart. Phys. **29**, 386 (2008)
30. Capozziello, S., De Filippis, E., Salzano, V.: Mon. Not. R. Astron. Soc. **394**, 947 (2009)
31. Pogosian, L., Silvestri, A.: Phys. Rev. D **77**, 023503 (2008)
32. Capozziello, S., Tsujikawa, S.: Phys. Rev. D **77**, 10750 (2008)
33. Capozziello, S., De Laurentis, M., Nojiri, S., Odintsov, S.D.: Gen. Relativ. Gravit. **41**, 2313 (2009)
34. Stelle, K.S.: Gen. Relativ. Gravit. **9**, 343 (1978)
35. Nishizawa, A., Taruya, A., Hayama, K., Kawamura, S., Sakagami, M.: Phys. Rev. D **79**, 082002 (2009)
36. Bogdanos, C., Capozziello, S., De Laurentis, M., Nesseris, S.: arXiv:0911.3094 [gr-qc] (2009)
37. Magnano, G., Sokolowski, L.M.: Phys. Rev. D **50**, 5039 (1994)
38. Faraoni, V., Gunzig, E.: Int. J. Theor. Phys. **38**, 217 (1999)
39. Maggiore, M.: Phys. Rep. **331**, 283 (2000)
40. Will, C.M.: Living Rev. Relativ. **9**, 3 (2006) (http://www.livingreviews.org/lrr-2006-3 or gr-qc/0510072)
41. Capozziello, S., Stabile, A., Troisi, A.: Phys. Rev. D **76**, 104019 (2007)
42. Capozziello, S., Stabile, A.: Class. Quantum Gravity **26**, 085019 (2009)
43. Capozziello, S., Stabile, A., Troisi, A.: Class. Quantum Gravity **25**, 085004 (2008)
44. Landau, L., Lifshitz, E.M.: Field Theory. Pergamon Press, Elmsford (1973)
45. Multamaki, T., Putaja, A., Vagenas, E.C., Vilja, I.: Class. Quantum Gravity **25**, 075017 (2008)
46. Starobinsky, A.A.: Phys. Lett. B **91**, 99 (1980)
47. Kerner, R.: Gen. Relativ. Gravit. **14**, 453 (1982)
48. Bellucci, S., Capozziello, S., De Laurentis, M., Faraoni, V.: Phys. Rev. D **79**, 104004 (2009)
49. Capozziello, S., De Laurentis, M., Francaviglia, M.: Astropart. Phys. **29**, 125 (2008)